On a certain class of cubic surfaces related to the Simson–Wallace theorem

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Outline of the talk:

- Introduction
- Extension of S–W theorem on skew quadrilaterals
- Properties of the S–W locus:
  - Decomposibility
  - Structure of lines on the cubic
  - Singular cases
- Final remarks

Introduction

3D extension of the well-known Simson–Wallace theorem on a tetrahedron [Roanes, 2000], [Pech, 2005] reads:

Let K, L, M, N be orthogonal projections of the point P to the faces BCD, ACD, ABD, ABC of a tetrahedron $ABCD$. Then the locus of P such that the tetrahedron $KLMN$ has a constant volume s is the cubic surface

$$G := ac^2f^3E + sQ = 0,$$

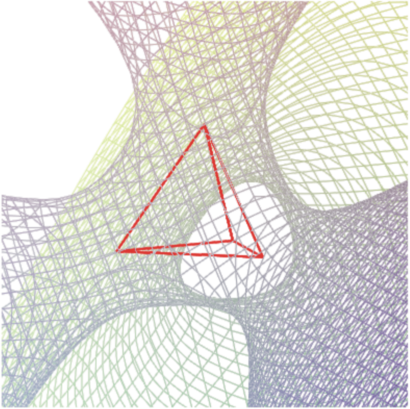
where

$$E = c^2f^2p^2q + cf(e^2+f^2-ce)p^2r + cf^2(a-2b)pq^2 + cf^2(a-2d)pr^2 + 2cef(b-d)pqr + b(b-a)f^2q^3 + f(be(a-b)+cd(d-a)+cf^2)q^2r + f^2(b^2-ab+c^2-2ce)qr^2 + (be(a-b)+cd(d-a)+ce(e-c))fr^3 - ac^2f^2pq + acf(ce-e^2-f^2)pr + abcf^2q^2 + (a(c^2d-2bce+be^2)-(cd-be)^2+f^2(ab-b^2-c^2))fqr + (ce^2(ab+ad-2bd)+c^2de(d-a)+be^3(b-a)+f^2(a(cd-be)+e(b^2+c^2)))r^2$$

Introduction

and

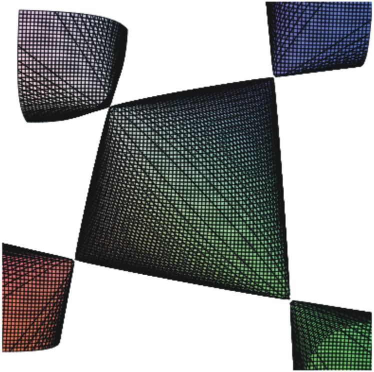
$$Q = 6(e^2 + f^2)((cd - be)^2 + f^2(b^2 + c^2))((c(a - d) - e(a - b))^2 + f^2((a - b)^2 + c^2)).$$



Regular tetrahedron, volume of $KLMN$ equals $-1/10$

Introduction

For $s = 0$ we obtain the famous Cayley cubic, with four singular points at the vertices of the corresponding tetrahedron $ABCD$.



Cayley cubic $4pqr - (p + q + r - 1)^2 = 0$ for regular tetrahedron

For generalization of the S–W locus in d-dimensional projective-metric space, see [Pech, J. of Geometry, to appear]

Extension of S–W theorem on skew quadrilaterals

The following is generalization of S–W theorem on skew quadrilaterals [Pech 2005]:

Theorem 1

Let K, L, M, N be orthogonal projections of a point P onto the sides AB, BC, CD, AD of a skew quadrilateral $ABCD$ respectively. Let $A = (0,0,0)$, $B = (a,0,0)$, $C = (b,c,0)$ and $D = (d,e,f)$. Then the locus of $P = (p,q,r)$ such that the tetrahedron $KLMN$ has a constant volume s is a cubic surface F

$$F := cfH + sR = 0, \tag{1}$$

where

$$R = 6(d^2 + e^2 + f^2)((b-d)^2 + (c-e)^2 + f^2)((a-b)^2 + c^2)$$

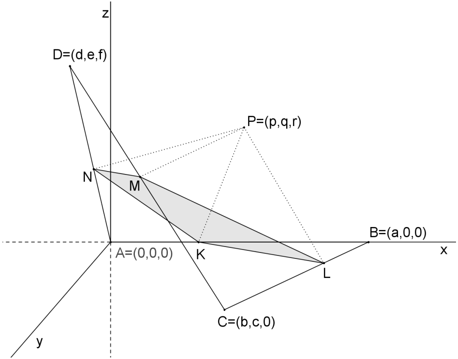
and

Extension of S–W theorem on skew quadrilaterals

$$H = p^3(c^2d(d-a)-(be^2+bf^2-2cde)(a-b))-p^2qc(ae(c-e)+f^2(a-2b))-p^2rcf(ac-2cd-2e(a-b))+pq^2(c^2(d^2+f^2-ad)+e(2cd-be)(a-b))+2pqrf(cd-be)(a-b)-pr^2f^2(ab-b^2-c^2)-q^3ace(c-e)-q^2racf(c-2e)+qr^2acf^2+p^2(cd(a^2(c-2e)+e(ab+b^2+c^2))+(e^2+f^2)(ab+b^2+c^2)(a-b)-c(e^2+f^2+d^2)(cd+ae-be))+pq(cd(d-a)(ab-b^2-c^2-ad+bd)-de(a-b)(b^2+c^2)+a^2ce(c-e)-cf^2(ab+b^2+c^2-a^2)+(a-b)((e^2+f^2)(be-cd)+bd^2e))-prf((ab-b^2-c^2)(bd+ce-d^2-e^2-f^2)-ac(2be+ac-2cd-2ae))+q^2ae(c(bd+ce-d^2-e^2-f^2)-(c-e)(ab-b^2-c^2))+qra(cf(bd+ce-d^2-e^2-f^2)-f(c-2e)(ab-b^2-c^2))+r^2af^2(ab-b^2-c^2)-pa(cd(c(ad-d^2+ce)-(be+de)(a-b))+(e^2+f^2)((b^2+c^2-ce)(a-b)-c^2d))+(qe+rf)a(bd+ce-d^2-e^2-f^2)(ab-b^2-c^2).$$

Extension of S–W theorem on skew quadrilaterals

Outline of the proof: Let $A = (0,0,0)$, $B = (a,0,0)$, $C = (b,c,0)$ and $D = (d,e,f)$. Suppose that $acf \neq 0$ since otherwise the quadrilateral is planar. Denote $K = (k_1,0,0)$, $L = (l_1,l_2,0)$, $M = (m_1,m_2,m_3)$, $N = (n_1,n_2,n_3)$ and $P = (p,q,r)$.



Extension of S–W theorem on skew quadrilaterals

Then

- $PK \perp AB \Leftrightarrow h_1 := a(p - k_1) = 0,$
- $L \in BC \Leftrightarrow h_2 := l_2(b - a) - c(l_1 - a) = 0,$
- $PL \perp BC \Leftrightarrow h_3 := (p - l_1)(b - a) + c(q - l_2) = 0,$
- $M \in CD \Leftrightarrow h_4 := (d - b)(m_2 - c) - (e - c)(m_1 - b) = 0,$

  $h_5 := (e - c)m_3 - (m_2 - c)f = 0,$

  $h_6 := (m_1 - b)f - m_3(d - b) = 0,$

Extension of S–W theorem on skew quadrilaterals

- $PM \perp CD \Leftrightarrow$

  $h_7 := (p - m_1)(d - b) + (q - m_2)(e - c) + (r - m_3)f = 0,$

- $N \in DA \Leftrightarrow h_8 := dn_2 - en_1 = 0, \; h_9 := dn_3 - fn_1 = 0,$

  $h_{10} := fn_2 - en_3 = 0,$

- $PN \perp DA \Leftrightarrow h_{11} := (p - n_1)d + (q - n_2)e + (r - n_3)f = 0,$

- Volume $KLMN = s \;\Leftrightarrow$

$$h_{12} := \begin{vmatrix} k_1, & 0, & 0, & 1 \\ l_1, & l_2, & 0 & 1 \\ m_1, & m_2, & m_3, & 1 \\ n_1, & n_2, & n_3, & 1 \end{vmatrix} - 6s = 0.$$

Extension of S–W theorem on skew quadrilaterals

- Elimination of $k_1, \ldots, n_3$ in the system $h_1 = 0$, $h_2 = 0$, $\ldots$, $h_{12} = 0$ yields the equation (1).[1]
- We see that $F = 0$ describes a cubic surface.
- Hence $P \in F$ is the necessary condition for the feet K, L, M, N to be coplanar.

Similarly, with the use of the program Epsilon, we can prove that $P \in F$ is the sufficient condition [Pech 2015]. $\square$

[1] We use software CoCoA which is freely distributed at `http://cocoa.dima.unige.it` and Epsilon library which is freely distributed at `http://www-calfor.lip6.fr/~wang/epsilon/`

Extension of S–W theorem on skew quadrilaterals

We can also proceed in another way to find F. Expressing $k_1, \ldots, n_3$ from the system above we get:

$$k_1 = p,$$

$$l_1 = (p(a-b)^2 + qc(b-a) + ac^2)/((a-b)^2 + c^2),$$

$$l_2 = (pc(b-a) + c^2 q + ac(a-b))/((a-b)^2 + c^2),$$

$$m_1 = (p(b-d)^2 + q(b-d)(c-e) + rf(d-b) + c(cd - be - de) + b(e^2 + f^2))/((b-d)^2 + (c-e)^2 + f^2),$$

$$m_2 = (p(b-d)(c-e) + q(c-e)^2 + fr(e-c) - bcd + cd^2 + b^2 e - bde + cf^2)/((b-d)^2 + (c-e)^2 + f^2),$$

$$m_3 = (pf(d-b) + qf(e-c) + f^2 r + f(b^2 + c^2 - bd - ce))/((b-d)^2 + (c-e)^2 + f^2),$$

Extension of S–W theorem on skew quadrilaterals

$$n_1 = (d^2p + deq + dfr)/(d^2 + e^2 + f^2),$$

$$n_2 = (dep + e^2q + efr)/(d^2 + e^2 + f^2),$$

$$n_3 = (dfp + efq + f^2r)/(d^2 + e^2 + f^2).$$

Substitution for $k_1, l_1, l_2, \ldots, n_3$ into

$$\begin{vmatrix} l_1 - k_1, & l_2, & 0 \\ m_1 - k_1, & m_2, & m_3 \\ n_1 - k_1, & n_2, & n_3 \end{vmatrix} = 0$$

gives H in the basic formula (1) in the shorter form

Extension of S–W theorem on skew quadrilaterals

$$H = c(dp+eq+fr)(p(d-b)+(e-c)q+fr-(d-b)d-(e-c)e-f^2)(cp+q(a-b)-ac)+(p(b-a)+cq+a(a-b))((-p(e^2+f^2)+qde+rdf)(p(d-b)+q(e-c)+rf+b^2+c^2-bd-ce)+(pd+qe+rf)(p((c-e)^2+f^2)-q(b-d)(c-e)-rf(d-b)-c(cd-be-de)-b(e^2+f^2))).$$

Later we will express F even in the more concise form.

In the following suppose that $s = 0$ in the formula

$$F = cfH + sR,$$

i.e. K, L, M, N are coplanar. Then $F = H$, since $cf \neq 0$.

Properties of the S–W locus

In this section some properties of the cubic which is associated with a skew quadrilateral $ABCD$ are investigated.

Particularly the following properties of the cubic H are studied:

- decomposability,
- structure of lines on the cubic,
- singular cases.

Properties of the locus — decomposability

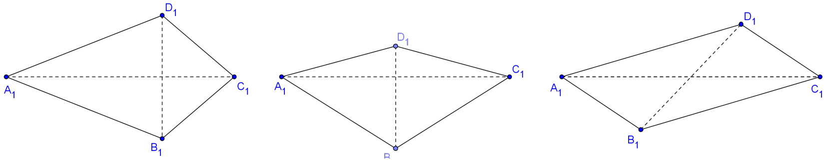
The next theorem is on decomposability of the S–W locus.

Theorem 2

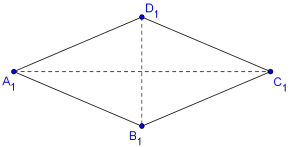
The cubic surface which is associate with a skew quadrilateral *ABCD* is decomposable iff two pairs of sides — either adjacent or opposite — of *ABCD* are of equal lengths p, q.
If $p \neq q$ the cubic decomposes into a plane and a one–sheet hyperboloid,
if $p = q$, i.e., if *ABCD* is equilateral, the cubic decomposes into three mutually orthogonal planes.

In the next figures you shall see horizontal views of quadrilaterals *ABCD* when the cubic is decomposable.

Properties of the locus — decomposability



Horizontal view of $ABCD$ onto the plane parallel to diagonals AC and BD — two deltoids and a parallelogram.

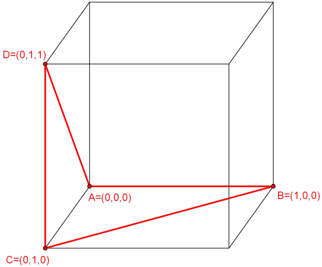


Rhombus — all sides of $ABCD$ are of equal lengths.

Properties of the S–W locus — decomposability

Example 1

For $a = 1,\ b = 0,\ c = 1,\ d = 0,\ e = 1,\ f = 1$

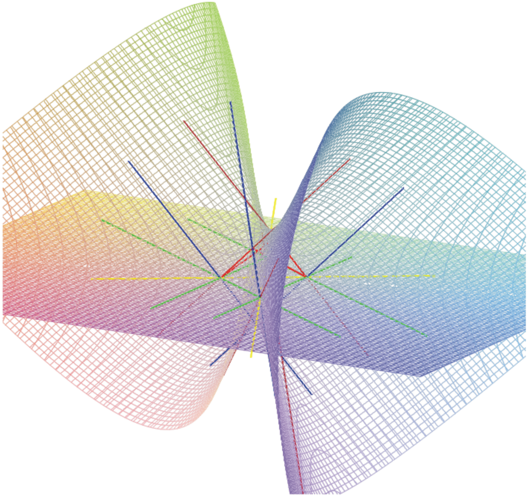


we get the cubic

$$(pq - q^2 - pr - qr + q + r)(p + r - 1) = 0,$$

which decomposes into a plane and hyperboloid.

Properties of the S–W locus — decomposability



Note that two pairs of opposite sides of $ABCD$ are of equal lengths.

Properties of the S–W locus — structure of lines

The well-known Salmon–Cayley theorem states that a smooth cubic surface over algebraic closed field contains exactly 27 lines. In the following the number of real lines which lie on the cubic H is investigated.

Planes $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and A_8 which are perpendicular to the sides of $ABCD$ and pass through its vertices are crucial for investigation of the structure of lines on the cubic:

$$A_1: \ A_1 \perp DA, \ D \in A_1, \qquad A_5: \ A_5 \perp BC, \ B \in A_5,$$
$$A_2: \ A_2 \perp DA, \ A \in A_2, \qquad A_6: \ A_6 \perp BC, \ C \in A_6,$$
$$A_3: \ A_3 \perp CD, \ C \in A_3, \qquad A_7: \ A_7 \perp AB, \ A \in A_7,$$
$$A_4: \ A_4 \perp CD, \ D \in A_4, \qquad A_8: \ A_8 \perp AB, \ B \in A_8.$$

The planes belong to the system of tritangent planes which intersect the cubic H in three lines.

Properties of the S–W locus — structure of lines

We can easily verify that it (surprisingly) holds

$$H = A_1A_3A_5A_7 - A_2A_4A_6A_8, \tag{2}$$

or

$$H = (dp+eq+fr-d^2-e^2-f^2)((d-b)p+(e-c)q+fr-(d-b)b-(e-c)c)((b-a)p+cq-(b-a)a)p-(dp+eq+fr)((d-b)p+(e-c)q+fr-(d-b)d-(e-c)e-f^2)((b-a)p+cq-(b-a)b-c^2)(p-a).$$

This is the most concise form of H that I have found.

The importance of (2) appears by searching for lines lying on the cubic. Namely from $H = 0$ and (2) we get that the line $A_i \cap A_j$, $i = 1, 3, 5, 7,\ j = 2, 4, 6, 8$ belongs to H.

Properties of the S–W locus — structure of lines

From (2) we obtain the following 12 lines which belong to the cubic surface:

$$a = A_2 \cap A_7,\ b = A_8 \cap A_5,\ c = A_6 \cap A_3,\ d = A_4 \cap A_1,$$
$$e = A_2 \cap A_5,\ f = A_8 \cap A_3,\ g = A_6 \cap A_1,\ h = A_4 \cap A_7,$$
$$i = A_7 \cap A_6,\ j = A_2 \cap A_3,\ k = A_8 \cap A_1,\ l = A_5 \cap A_4.$$

Another 6 tritangent planes given by pairs of parallel lines:

$$A_9 = a \cup k, \quad A_{10} = b \cup i, \quad A_{11} = c \cup l,$$
$$A_{12} = d \cup j, \quad A_{13} = e \cup g, \quad A_{14} = f \cup h.$$

Properties of the S–W locus — structure of lines

Denote:

$$C_1 = ab - bd - ce,$$

$$C_2 = b^2 + c^2 - ab - bd - ce,$$

$$C_3 = d^2 + e^2 + f^2 - a^2 + ab - bd - ce.$$

If $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$ then we obtain another three lines m, n, o

$$m = A_{10} \cap A_{12}, \quad n = A_9 \cap A_{11}, \quad o = A_{13} \cap A_{14}$$

which belong to the cubic H.

It holds:

$$
\begin{aligned}
a)\ &\text{The lines } m, n \text{ coincide} &\Leftrightarrow\quad & C_1 = 0 \quad \text{and} \quad C_2 \neq 0, C_3 \neq 0.\\
b)\ &\text{The lines } m, o \text{ coincide} &\Leftrightarrow\quad & C_2 = 0 \quad \text{and} \quad C_1 \neq 0, C_3 \neq 0.\\
c)\ &\text{The lines } n, o \text{ coincide} &\Leftrightarrow\quad & C_3 = 0 \quad \text{and} \quad C_1 \neq 0, C_2 \neq 0.
\end{aligned}
$$

Finally we add the plane

$$A_{15} = m \cup n \cup o.$$

Note that A_{15} passes through the center S of the circumsphere of $ABCD$.

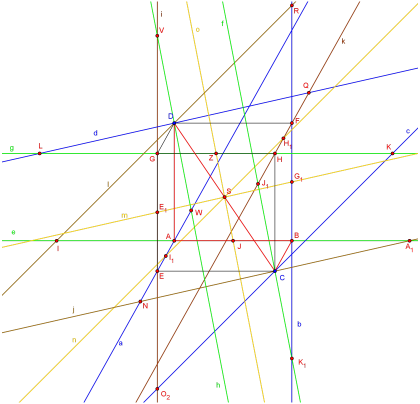
Properties of the S–W locus — structure of lines

Example 2

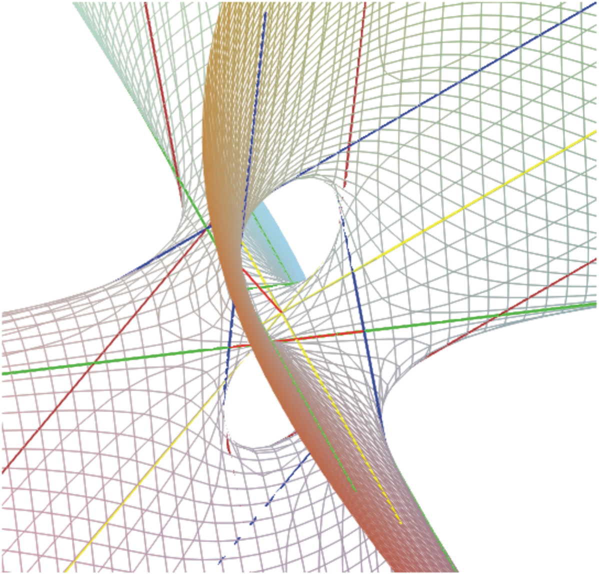
For $a = 1,\ b = 1,\ c = 1,\ d = 0,\ e = 0,\ f = 1$ we get the cubic

$$p^2q + pq^2 - p^2r - q^2r + pr^2 + qr^2 - 2pq - r^2 + r = 0$$

which contains 15 lines.



Properties of the S–W locus — structure of lines



Cubic $p^2q + pq^2 - p^2r - q^2r + pr^2 + qr^2 - 2pq - r^2 + r = 0$

Properties of the S–W locus — structure of lines

The planes above yield the following 10 canonical forms[2] of the cubic H:

$$
\begin{aligned}
H &= A_2A_4A_{10} + A_5A_7A_{12}, & H &= A_1A_3A_{10} + A_6A_8A_{12},\\
H &= A_4A_8A_{13} + A_1A_5A_{14}, & H &= A_3A_7A_{13} + A_2A_6A_{14}\\
H &= A_1A_7A_{11} + A_4A_6A_9, & H &= A_2A_8A_{11} + A_3A_5A_9,\\
H &= A_1A_2A_{15} + A_9A_{12}A_{13}, & H &= A_3A_4A_{15} + A_{11}A_{12}A_{14},\\
H &= A_5A_6A_{15} + A_{10}A_{11}A_{13}, & H &= A_7A_8A_{15} + A_9A_{10}A_{14}.
\end{aligned}
$$

[2] The cubic H is expressed in a canonical form if $H = abc + def$, where a, b, c, d, e, f are linear factors.

27 lines on the cubic

So far we have investigated cubics H which contain 15 lines. Is there a case when a cubic H contains 27 real lines?

The answer gives the following theorem:

Theorem 3
Let $C_1 \neq 0,\ C_2 \neq 0,\ C_3 \neq 0$. Then a cubic H contains exactly 27 distinct real lines iff

$$(C_1 C_2 - C_2 C_3 + C_3 C_1)^2 - 4a^2 C_1 C_2 C_3 \ > \ 0\ . \qquad (3)$$

27 lines on the cubic

Example 3

For a skew quadrilateral $a = 1$, $b = -2$, $c = 1$, $d = 2$, $e = -1$, $f = 1$ we get the cubic

$$2p^3 - 3p^2q - 3pq^2 + 2q^3 - 3p^2r - 3q^2r + 7pr^2 + qr^2 + 24p^2 + 24pq - 3q^2 - 74pr + 10qr - 7r^2 - 26p - 77q + 77r = 0.$$

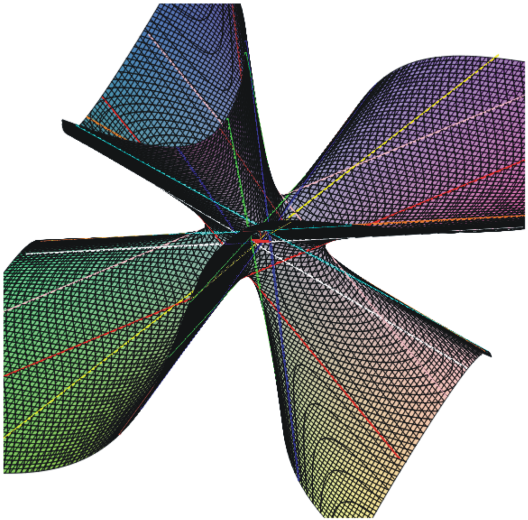
It holds $C_1 = 3, C_2 = 12, C_3 = 8$

and

$$(C_1C_2 - C_2C_3 + C_3C_1)^2 - 4a^2C_1C_2C_3 = 144 \; > \; 0 \; .$$

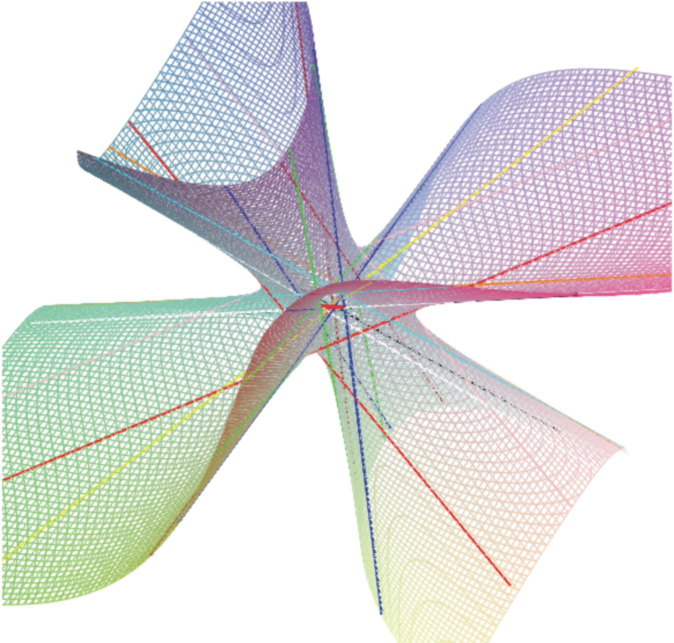
Then by the Theorem 3 there exist 27 real lines on the cubic.

27 lines on the cubic

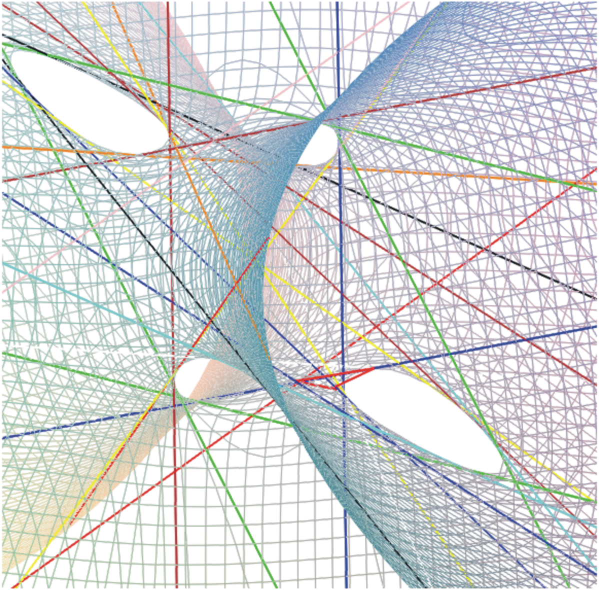


The cubic $2p^3 - 3p^2q - 3pq^2 + 2q^3 - 3p^2r - 3q^2r + 7pr^2 + qr^2 + 24p^2 + 24pq - 3q^2 - 74pr + 10qr - 7r^2 - 26p - 77q + 77r = 0$ contains 27 real lines

27 lines on the cubic



27 lines on the cubic



27 lines on the cubic

Example 2 revisited

For $a = 1,\ b = 1,\ c = 1,\ d = 0,\ e = 0,\ f = 1$ we get the cubic

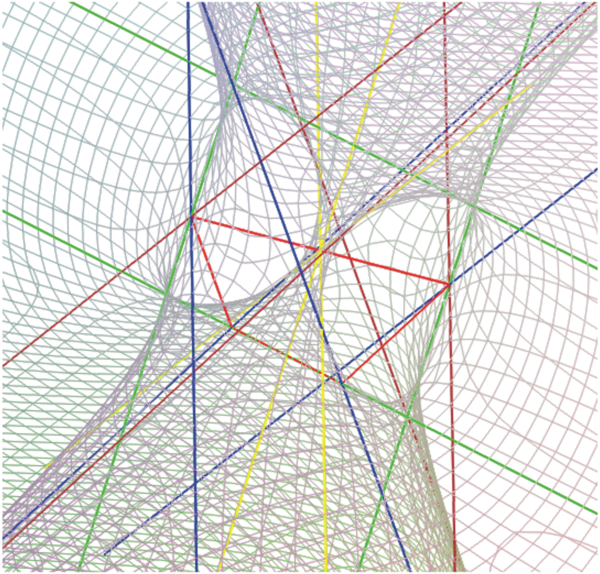
$$p^2q + pq^2 - p^2r - q^2r + pr^2 + qr^2 - 2pq - r^2 + r = 0$$

with $C_1 = C_2 = C_3 = 1$, and

$$(C_1C_2 - C_2C_3 + C_3C_1)^2 - 4a^2C_1C_2C_3 = -3.$$

Thus the cubic contains exactly 15 real lines.

27 lines on the cubic



Properties of the S–W locus — singular cases

Theorem 4

Let $C_1 = 0,\ C_2 C_3 \neq 0$ or $C_2 = 0,\ C_1 C_3 \neq 0$ or $C_3 = 0,\ C_1 C_2 \neq 0$. Then H possesses 2 singular points.

Outline of the proof: Let $C_1 = 0,\ C_2 \neq 0,\ C_3 \neq 0$. Then the lines m and n coincide and the planes A_9, A_{10}, A_{11} and A_{12} have the common line $m = n$.

Since $A_9 = a \cup k$, $A_{10} = b \cup i$, $A_{11} = c \cup l$ and $A_{12} = d \cup j$, then the lines a, k, b, i, c, l, d, j intersect the common line $m = n$. It is easy to verify that the lines a, c, i, j, m meet at

$$S_1 = \Big[0, \frac{b^2 + c^2 - ab}{c}, \frac{e(ab - b^2 - c^2)}{cf}\Big],$$

Properties of the S–W locus — singular cases

and the lines b, d, k, l, m at

$$S_2 = \left[a, 0, \frac{d^2 + e^2 + f^2 - ad}{f}\right].$$

Similarly we proceed if $C_2 = 0,\ C_1 C_3 \neq 0$ or $C_3 = 0,\ C_1 C_2 \neq 0$.

Remark: How to find singular points of H classically?

Solving the system

$\{H = 0, \frac{\partial H}{\partial p} = 0, \frac{\partial H}{\partial q} = 0, \frac{\partial H}{\partial r} = 0\}$

together with one condition $C_i = 0,\ i = 1, 2, 3$, is feasible in concrete cases.

In general it does not give any result at the moment.

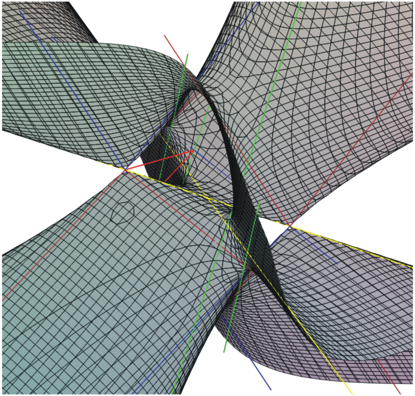
Properties of the S–W locus — singular cases

Example 4

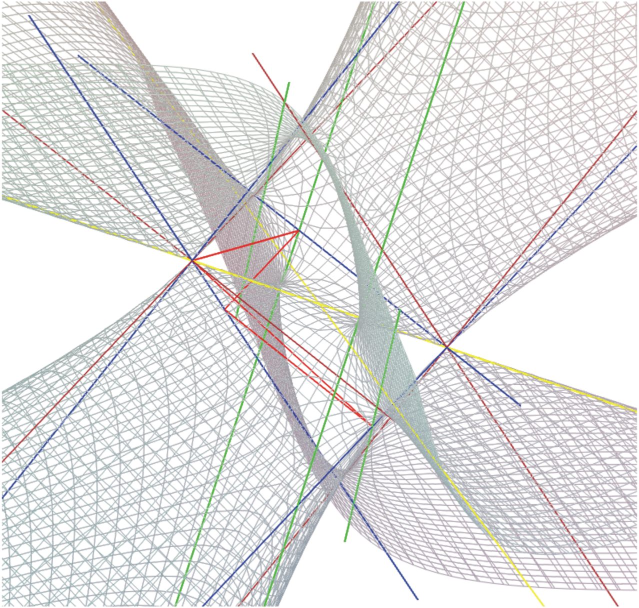
For $a = 1,\ b = 0,\ c = 1,\ d = 0,\ e = 0,\ f = 2$ we get the cubic

$$2p^2q - 2pq^2 + p^2r + q^2r - 2pr^2 - 2qr^2 - 2p^2 + 3pr + 3qr + 2r^2 + 2p - 4r = 0,$$

where $C_1 = 0,\ C_2 = 1,\ C_3 = 3$. 2 singular points $(0, 1, 0)$, $(1, 0, 2)$.



Extension of S–W theorem on skew quadrilaterals



Extension of S–W theorem on skew quadrilaterals

Example 5

For $a = 1, b = 1, c = 1/2, d = 0, e = 0, f = 1$ we get the cubic

$$4p^2q + 2pq^2 - 2p^2r - 2q^2r + 2pr^2 + 4qr^2 - 5pq - 3qr - 2r^2 + 2r = 0$$

with

$C_1 = 1, C_2 = \frac{1}{4}, C_3 = 1$

and

$$(C_1C_2 - C_2C_3 + C_3C_1)^2 - 4a^2C_1C_2C_3 = 0.$$

Properties of the S–W locus — singular cases

What is geometric meaning of relations

$$C_1 := ab - bd - ce = 0,$$

$$C_2 := b^2 + c^2 - ab - bd - ce = 0,$$

$$C_3 := d^2 + e^2 + f^2 - a^2 + ab - bd - ce = 0 \ ?$$

It holds:

$$C_1 = 0 \ \Leftrightarrow \ AC \perp BD,$$

$$C_2 = 0 \ \Leftrightarrow \ (C - A) \perp (\tfrac{A+C}{2} - \tfrac{B+D}{2}),$$

$$C_3 = 0 \ \Leftrightarrow \ (D - B) \perp (\tfrac{A+C}{2} - \tfrac{B+D}{2}).$$

The last two conditions mean that the line connecting the centers of AC and BD is othogonal either to AC or to BD.

Final remarks

- Instead of a skew quadrilateral A, B, C, D we can study four arbitrary lines a, b, c, d. We made first steps in searching for the locus of P such that its orthogonal projections onto the lines a, b, c, d are coplanar.
- The problem is quite complex, a special case is when the lines a, b, c, d form a skew quadrilateral.
- According to the mutual position of lines a, b, c, d we get both quadrics and cubics.

Final remarks

- Is it possible to get all cubics?
- Which kinds of quadrics can we obtain?
- The class of cubics H which are associated with a skew quadrilateral contains various kinds of cubics as we could see. It seems that the class of cubics H is sufficiently rich that it could serve as a model for demonstration of some types of cubic surfaces.

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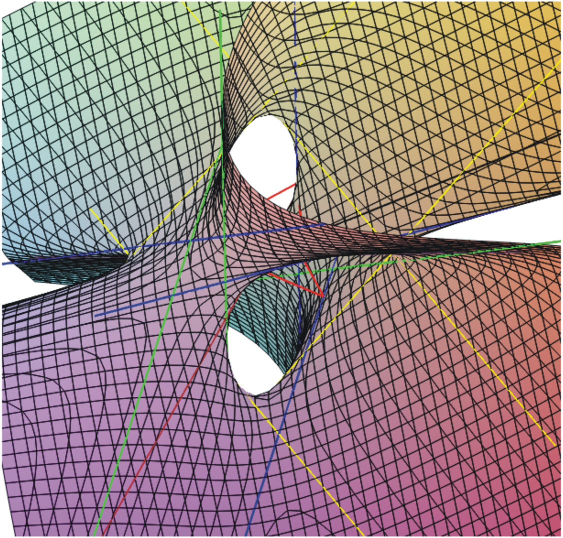

Thank you for your attention