Dependence of Axioms for Weak Geometries Proved Syntactically

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- Work done with Jesse Alama and his aggregate of automatic theorem provers and finite counterexample searchers `Tipi`.

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- we get $\mathcal{A}'$, the universal absolute theory of σ, μ, and L.

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- $\mathcal{A}_1 = \mathcal{A}+$ (A16) is thus axiomatized by (A2), (A3), (A4), (A16).

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- $\mathcal{A}_1 \subsetneq \mathcal{A}_2$, given that the point-reflection and midpoint operations of the hyperbolic plane do satisfy $\mathcal{A}_1$, but not $\mathcal{A}_2$.
- We do not know whether $\mathcal{A}_2 \subsetneq \mathcal{V}$, and `Tipi` could not help. In case our conjecture holds, the above is true, given that all axioms of $\mathcal{A}_2$ are at most 3-variable statements.

More intermediate geometries

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- It states that the medians from b to $\mu(ac)$ and from a to $\mu(bc)$ meet in the point o, which divides $a\mu(bc)$ and $b\mu(ac)$ in the ratio 2 : 1 (vertex: midpoint). Here a stands for $\sigma(\sigma(o\mu(bc))o)$. Of course, the picturesque geometric statement we provided should be taken with a grain of salt, as there is no mention of the fact that the vertices of our triangle are not collinear.

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- with $\mathcal{A}_3 = \mathcal{A}+$ (A18), we have $\mathcal{A} \subsetneq \mathcal{A}_1 \subsetneq \mathcal{A}_2 \subset \mathcal{V}$ and $\mathcal{A} \subsetneq \mathcal{A}_3 \subsetneq \mathcal{A}_2$

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- $\mathcal{A}'^{+}$ denotes the theory obtained by adding (A16), (A20), and (A21) to $\mathcal{A}'$. $\mathcal{A}'^{+}$ is the richest absolute L, σ, and μ-based universal theory we consider.

Maria Teresa Calapso's axiom

- An axiom that can be stated using L and μ, and which is easily seen to follow from (A8)-A(10) and A(17) (first considered by M. T. Calapso (1971), and shown to be equivalent to the rectangle axiom inside the theory of metric planes by R. Struve and V. Pambuccian (2009)) states that the vertex a, the midpoint of the opposite side $\mu(bc)$, and the midpoint of the midline $\mu(ab)\mu(ac)$ are collinear, i. e.

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- (A22) $L(a\mu(bc)\mu(\mu(ab)\mu(ac)))$

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- (A22) $L(a\mu(bc)\mu(\mu(ab)\mu(ac)))$
- It is clear that (A22) is not in $\mathcal{A}'^{+}$, given that the the hyperbolic plane with the usual point-reflection, midpoint, and collinearity notions is a model of $\mathcal{A}'$, but (A22) holds only for isosceles triangles.

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- Barbilian (1936): "The segments PA, PB, PC, joining a point P with the vertices of an equilateral triangle satisfy the generalized triangle inequality" $\Leftrightarrow$ "Sum of angles in a triangle is $\leq 180°$"

Lagrange (1806)

- In a proof of the Euclidean parallel postulate, in a paper read on 3 February 1806 at the *Institut de France*, Lagrange introduces an axiom which states that “If a and b are two parallels from P to g, then the reflection of a in b is parallel to g as well.”

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- Bachmann (1964) found another statement equivalent to the *Lotschnittaxiom*: "Through any point inside a right angle one can draw a line that intersects both sides of that angle."

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- Bachmann (1964) found another statement equivalent to the *Lotschnittaxiom*: "Through any point inside a right angle one can draw a line that intersects both sides of that angle."
- Pambuccian (1994), the universal statement: "In an isosceles triangle with base angles of 45°, the altitude to the base is smaller than the base."

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- Dehn (1900): The rectangle axiom is strictly weaker than the Euclidean parallel postulate.

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- Proof (Pambuccian(1994)) is not synthetic, as it uses Pejas's 1961 algebraic description of models of Hilbert's plane absolute geometry.

Szmielew's proof of the Pasch axiom from the Circle Axiom

- Is not synthetic. Thus one does not show how to construct the needed intersection point based on the opertaions present in her axiomatization: segmentntransport, line-circle intersection, line intersection.