From Hilbert to Tarski

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University of Strasbourg - ICube - CNRS

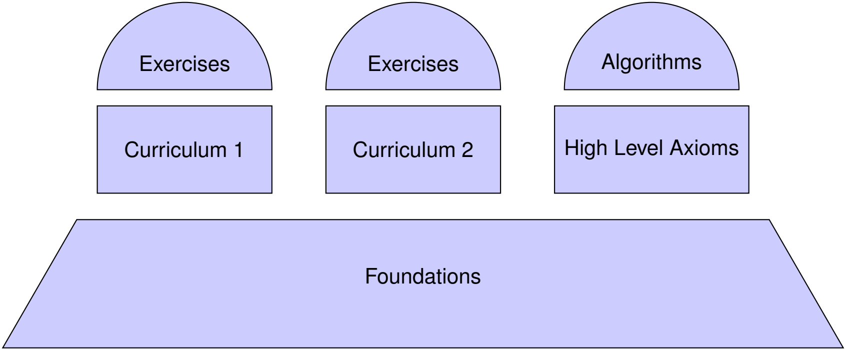
ADG 2016, Strasbourg

The project GeoCoq

A library of machine checked proofs in geometry.

Aimed Applications:

1. Education
2. Proof of computational geometry algorithms



Foundations of geometry

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- Synthetic approach: geometric objects and axioms about them.

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  - Euclid

Euclid
(325 av. J.-C. - 265 av. J.-C.)

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Euclide.
Les éléments.
Presses Universitaires de France, 1998.
Traduit par Bernard Vitrac.

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David Hilbert
(1862 - 1943)

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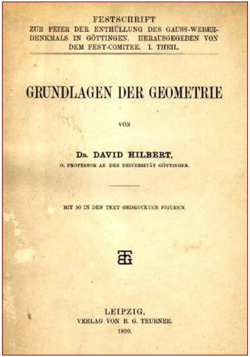

FESTSCHRIFT
ZUR FEIER DER ENTHÜLLUNG DES GAUSS-WEBER-DENKMALS IN GÖTTINGEN. HERAUSGEGEBEN VON DEM FEST-COMITEE. I. THEIL.

GRUNDLAGEN DER GEOMETRIE

VON

DR. DAVID HILBERT,
O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

MIT 50 IN DEN TEXT GEDRUCKTEN FIGUREN.

LEIPZIG,
VERLAG VON B. G. TEUBNER.
1899.

David Hilbert.

Foundations of Geometry (Grundlagen der Geometrie).
Open Court, La Salle, Illinois, 1960.
Second English edition, translated from the tenth German edition by Leo Unger. Original publication date, 1899.

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Alfred Tarski
(1901 - 1983)

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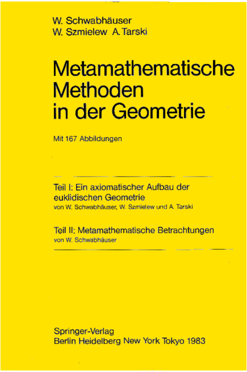

Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski.
Metamathematische Methoden in der Geometrie.
Springer-Verlag, Berlin, 1983.

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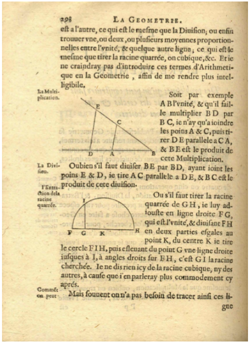

298 LA GEOMETRIE.

eſt a l'autre, ce qui eſt le meſme que la Diuiſion; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui eſt le meſme que tirer la racine quarrée, on cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligibile.

La Multiplication. Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diuiſion. Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

l'Extraction de la racine quarrée. Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Commét on peut Mais ſouuent on n'a pas beſoin de tracer ainſi ces lignes

René Descartes.

La Géométrie.
Leydle, 1637.

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George David Birkhoff
(1884 - 1944)

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George David Birkhoff.

A set of postulates for plane geometry (based on scale and protractors). *Annals of Mathematics*, 33, 1932.

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Felix Klein
(1849 - 1925)

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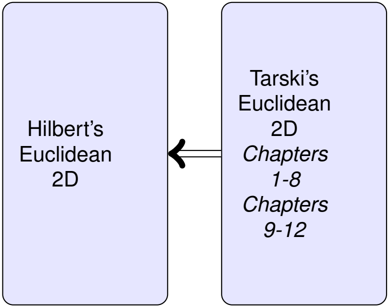
Felix C. Klein.
A comparative review of recent researches in geometry, 1872.

Previous work

Tarski's
Euclidean
2D
Chapters
1-8

Mechanical Theorem Proving in Tarski's geometry, ADG 2006

Previous work



Mechanical Theorem Proving in Tarski's geometry, ADG 2006
From Tarski to Hilbert, ADG 2012

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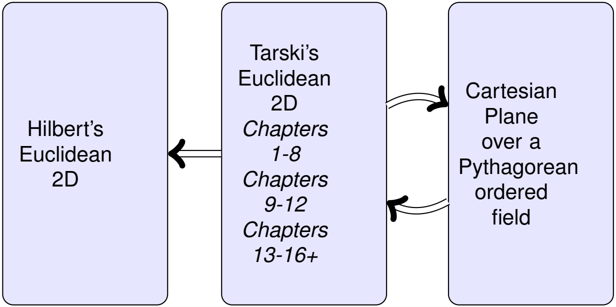

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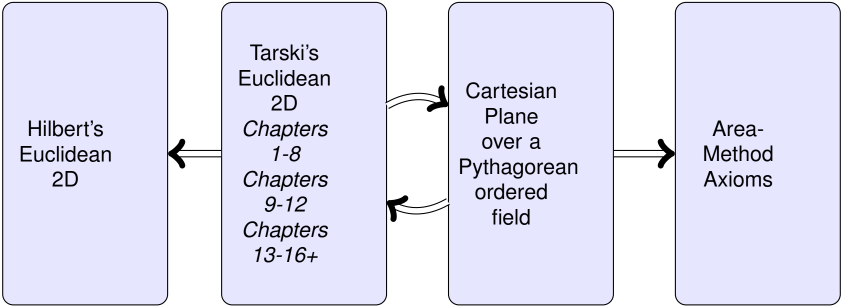

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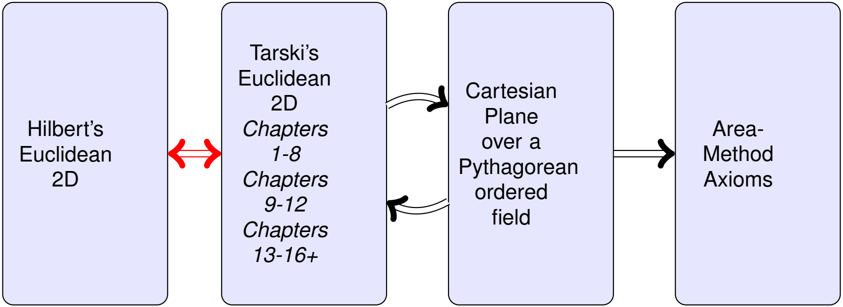

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From Tarski to Descartes, SCSS 2016
From Hilbert to Tarski, ADG 2016

- Formalization of Hilbert's *Foundations of Geometry*:
  - Isabelle/HOL Meikle and Fleuriot
  - Isabelle/HOL and HOL-Light Scott and Fleuriot
  - HOL-Light Richter
- Formalization of Tarski's Geometry:
  - Isabelle/HOL Petrović, Makarios (Euclidean and non-Euclidean model)
  - HOL-Light Richter
- Other formalizations of geometry in Coq: Duprat, Guilhot

Motivation

The problem

What are Hilbert's axioms?

- There are ten editions of the *Foundations of Geometry*.
- Hilbert's axioms are expressed in natural language: there is room for interpretation.

How to be sure that our formalization of the axioms is fine?

1. The axioms are not contradictory (there is a model): our ADG 2012 paper.
2. There are enough axioms to capture a set of geometric facts (descriptively complete): this presentation.

The usual argument

Tarski's axioms A_1-A_{10}	are bi-interpretable with the theory of Pythagorean ordered field.
Hilbert's axioms Group I-IV	are bi-interpretable with the theory of Pythagorean ordered field.

With this approach, the formalizations of both Hilbert's and Tarski-Schwabhäuser-Szmielew books are needed.
This argument tells nothing about the neutral geometry (A_1-A_9).

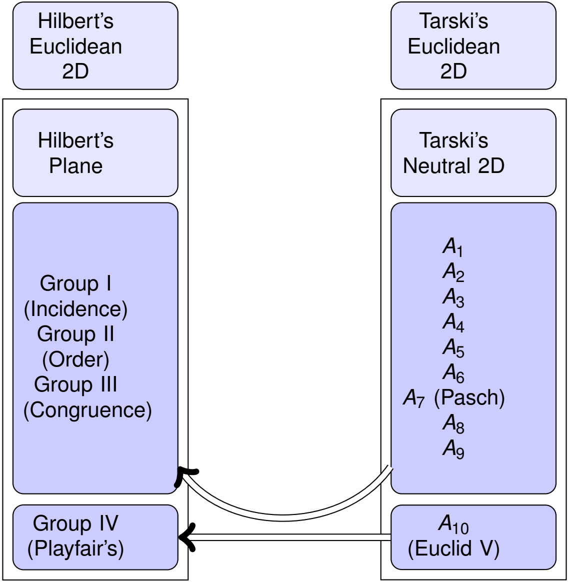
Our approach

From Tarski to Hilbert:

From Hilbert to Tarski:

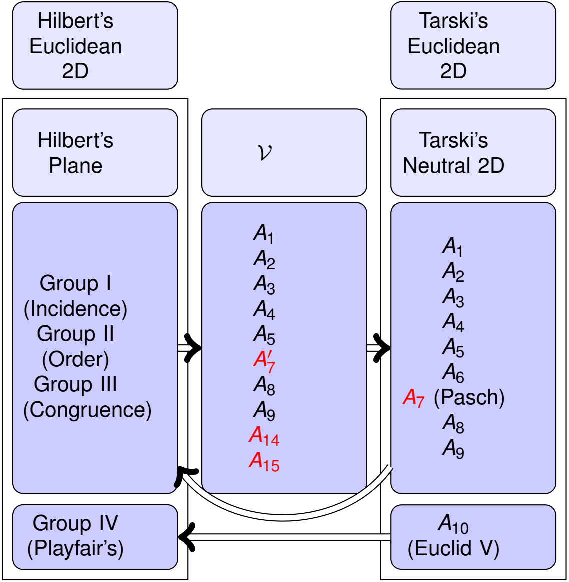
- an intermediate pier (*i.e.* an intermediate axiom system).
- two separate bridges

Overview



Hilbert's Euclidean 2D
Tarski's Euclidean 2D
Hilbert's Plane
Tarski's Neutral 2D
Group I (Incidence) Group II (Order) Group III (Congruence)
A_1 A_2 A_3 A_4 A_5 A_6 A_7 (Pasch) A_8 A_9
Group IV (Playfair's)
A_{10} (Euclid V)

Overview



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$\mathcal{V}$
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A_1 A_2 A_3 A_4 A_5 A_7' A_8 A_9 A_{14} A_{15}
A_1 A_2 A_3 A_4 A_5 A_6 A_7 (Pasch) A_8 A_9
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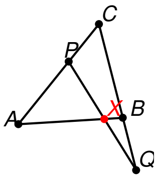
The Variant

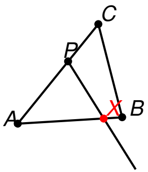
We separate the degenerate case of Pasch's axiom.

Pasch's axiom

Allows to formalize some gaps in Euclid's Elements.



Tarski's Pasch



Hilbert's Pasch

Tarski has a version expressed using betweenness and include degenerate cases:

$$Bet\ A\ P\ C \wedge Bet\ Q\ B\ C \Rightarrow \exists X, Bet\ P\ X\ Q \wedge Bet\ B\ X\ A$$

Hilbert has a version expressed using a disjunction and exclude the flat triangle.

A new formalization

Our 2012 formalization was correct but not descriptively complete.

The types of changes we made:

1. Remove useless (dependent) axioms
2. Add other axioms
3. Rephrase some axioms
4. Remove dependent types

Example of change of Type 1 (Remove Axioms)

- Given three collinear distinct points at least one of them is between the other two
- Uniqueness of segment construction
- Existence of parallel line

Examples of change of Type 2 (Adding Axioms)

Hilbert does not say explicitly that:

```
forall A B C, BetH A B C -> A<>C
```

Hilbert has a binary relation between segments, we have a quaternary relation between points, so we need:

```
forall A B C D , CongH A B C D -> CongH A B D C
```

Example of change of Type 3 (Rephrase Axiom)

Evil is in the details

Lower Dimension Axiom:

There exists three non collinear points.
Collinear := there exists a line such the three points belong to this line.

Problem

There are three non collinear points does no imply that they are distinct! There is a model of Group I-II with only one point and no lines.

Rephrased Lower Dimension Axiom:

There exists a point P_0 and line l_0 such that $P_0 \notin l_0$.

Change of Type 4 (Remove Dependent Types)

Don't open Pandora's box

Dependent type = data + proof
Faithful to Hilbert's presentation, but hard to manipulate in Coq.

Example

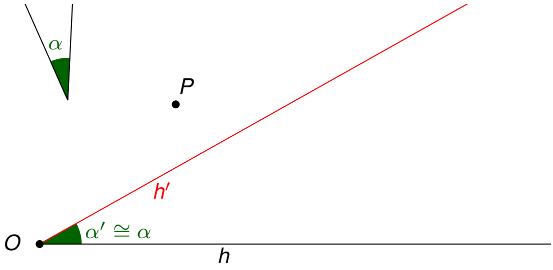
```
Record Triple {A:Type} : Type :=
 build_triple {V1 : A ;
 V : A ;
 V2 : A ;
 Pred : V1 <> V /\ V2 <> V}.

Definition angle := build_triple Point.
```

Example of change of Type 4-5

A simplification

Axiom (IV-4)

Given an angle α, a half-line h emanating from a point O and given a point P, not on the line generated by h, there is a unique half-line h′ emanating from O, such that the angle α' defined by (h, O, h') is congruent with α and such that every point inside α' and P are on the same side with respect to the line generated by h.



Example of change of Type 4-5

A simplification

In 2012 we had “a little bit verbose” axiom III 4 for existence and uniqueness of angle construction:

```
aux : forall (h h1 : Hline), P1 h = P1 h1 -> P2 h1 <> P1 h;
hcong_4_existence: forall a h P,
~Incid P (line_of_hline h) -> ~ BetH (V1 a)(V a)(V2 a) ->
exists h1, (P1 h) = (P1 h1) /\ (forall CondAux : P1 h = P1 h1,
CongaH a (angle (P2 h) (P1 h) (P2 h1) (conj (sym_not_equal (Cond h))
(aux h h1 CondAux))) /\
(forall M, ~ Incid M (line_of_hline h) /\ InAngleH (angle (P2 h) (P1 h) (P2 h1)
(conj (sym_not_equal (Cond h)) (aux h h1 CondAux))) M ->
same_side P M (line_of_hline h)));
hEq : relation Hline := fun h1 h2 => (P1 h1) = (P1 h2) /\
((P2 h1) = (P2 h2) \/ BetH (P1 h1) (P2 h2) (P2 h1) \/
BetH (P1 h1) (P2 h1) (P2 h2));
hline_construction a (h: Hline) P (hc:Hline) H :=
(P1 h) = (P1 hc) /\
CongaH a (angle (P2 h) (P1 h) (P2 hc) (conj (sym_not_equal (Cond h)) H)) /\
(forall M, InAngleH (angle (P2 h) (P1 h) (P2 hc)
(conj (sym_not_equal (Cond h)) H)) M ->
same_side P M (line_of_hline h));
hcong_4_unicity : forall a h P h1 h2 HH1 HH2,
~Incid P (line_of_hline h) -> ~ BetH (V1 a)(V a)(V2 a) ->
hline_construction a h P h1 HH1 -> hline_construction a h P h2 HH2 ->
hEq h1 h2
```

Example of change of Type 4-5

A simplification

The concept of “inside an angle” is not necessary + remove dependent types.

```
hcong_4_existence :
  forall A B C O X P,
   ~ ColH P O X -> ~ ColH A B C ->
   exists Y, CongaH A B C X O Y /\ same_side' P Y O X;
hcong_4_uniqueness :
  forall A B C O P X Y Y',
    ~ ColH P O X  -> ~ ColH A B C ->
    CongaH A B C X O Y -> CongaH A B C X O Y' ->
    same_side' P Y O X -> same_side' P Y' O X ->
    outH O Y Y'
```

The interpretation

Betweenness

Tarski's betweenness is not strict:

```
Definition Bet A B C :=
   BetH A B C \/ A = B \/ B = C.
```

Congruence

Hilbert's congruence tells nothing about degenerate segments:

```
Definition Cong A B C D :=
  (CongH A B C D /\ A <> B /\ C <> D) \/
  (A = B /\ C = D).
```

The proof: a summary

The not obvious axioms:

- **Five segments** Formalization of Hilbert's book (Theorems 12, 14, 15, 16, 17, 18) + non trivial degenerate case
- **Upper dimension** Our own proof
- **Parallel postulate** Previous work done in the context of Tarski's axioms

About constructive logic

- We assume decidability of point equality and incidence.
- We do not assume decidability of intersection of line for the equivalence between the neutral geometries.

```
forall P l, Incid P l  \/ ~ Incid P l;
forall A B,         A=B \/ ~ A=B;
```

Conclusion

Contributions

- We provide a formalization of Hilbert axioms with a formal proof that this version is correct and descriptively complete.
- The proof is about 5kloc of Coq.
- These results turn the GeoCoq library (a library about Tarski's geometry) into a library about foundations of geometry in general.

Conclusions

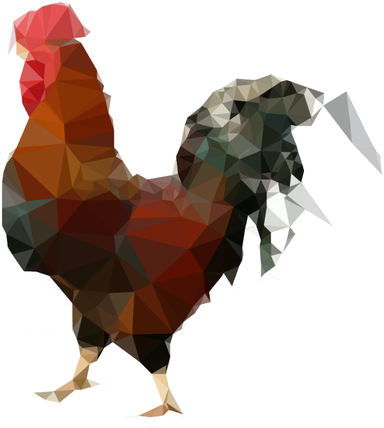
- Hilbert's axioms produce a lot of *administrative* work.
- It is better to keep the concepts of segments, rays and angles implicit.

Potential extensions

- Foundations based on group of transformations
- Generalization to nD

The full Coq development is available on github

`http://geocoq.github.io/GeoCoq/`



Questions ?