Formalization of a Surface Subdivision Allowing a Region with Holes without Coordinates

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Background

- A number of studies on symbolic treatment of geometric/topological properties
- Some of them use proof assistants on their formalization
  - Give a certified formalization, find drawbacks in the pen-and-paper proofs
- **Surface subdivision** is one of the topics of these studies

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Surface subdivision

- Embedding of a surface with a finite numbers of connected regions
- Related topics:
  - Graph embedding, four color theorem, Jordan's simple curve theorem
- Data structures:
  - Doubly connected edge list (e.g. [1])
  - Hypermap (e.g.[2])

[1] de Berg, M. et al., O.:"Computational Geometry, Springer-Verlag (1997).
[2] Dufourd, J.-F. and Bertot, Y.: "Formal study of plane Delaunay Triangulation." in *ITP 2010*, pp.211-226, LNCS, vol.6172, Springer-Verlag (2010).

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Surface subdivision

- Embedding of a surface with a finite numbers of co
- Re
  - s
- Da
  -
  - formalization using doubly connected edge list (e.g. [2])

Another formalization of a surface subdivision of the 2D plane

[1] Dufourd, J.~-F. and Bertot, Y.: "Formal study of plane Delaunay Triangulation." in *ITP 2010*, pp.211-226, LNCS, vol.6172, Springer-Verlag (2010).
[2] de Berg, M. et al., O.:"Computational Geometry, Springer-Verlag (1997).

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Outline

1. The target issue
2. A PLCA expression
3. Encoding PLCA in Coq
4. Representation of subdivision using PLCA
5. Conclusion

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2. A PLCA expression
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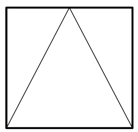
Surface subdivision (1)

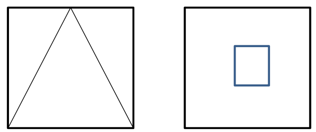
- Determining the locations of regions for a given set of points

without holes
(e.g. Delauney triangulation)



Surface subdivision (2)

- Determining the locations of regions for a given set of points

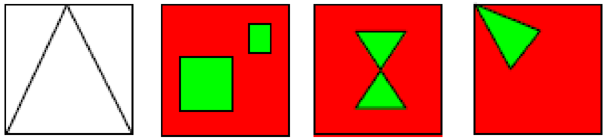


without holes
(e.g. Delauney triangulation)

+ with holes

- Disconnected components are regarded as regions embedded into the holes of another region

Examples of surface subdivisions



(a) (b) (c) (d)

a region with holes

a hole

(a) triangulation

(b) a region with holes

(c) a region with holes connected with a point

(d) a region with a hole connected to itself with a point

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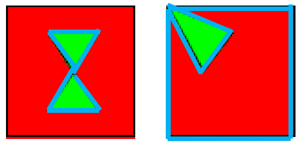
Border of regions



(c) (d)

The border of a region is not always a Jordan curve

a simple closed curve without a self-loop

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The target surface subdivision

- Consider a surface subdivision as a configuration in which both side of each line always belong to distinct regions

a representation of a figure showing its geometric or topological characteristics

- The configuration admits disconnected components

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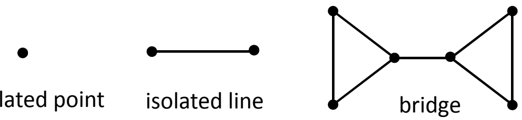
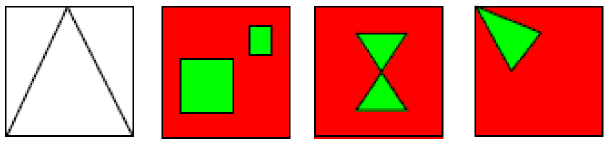
Allowed/disallowed configurations

Disallowed configurations



Allowed configurations



(a) (b) (c) (d)

The target surface subdivision

- Consider a surface subdivision as a configuration in which both side of each line always belong to distinct regions

  a representation of a figure showing its geometric or topological characteristics

- The configuration admits disconnected components

The idea

- Consider a surface subdivision as a configuration in which both side of each line always belong to distinct regions

  a representation of a figure showing its geometric or topological characteristics

- The configuration admits disconnected components

  ⇩

  PLCA planarity

The goal

- PLCA
  - Data structure in which all incidence relations between objects are stored [3]
  - Topological and geometric aspects can be distinguished
  - Not use coordinates
- Formalization with PLCA to represent a surface subdivision that allows disconnected components

[3] Takahashi, K. et al. "On embedding a qualitative representation in a two-dimensional plane." in *Spatial Cognition and Computation*, 8(1-2):4-26 (2008).

Outline

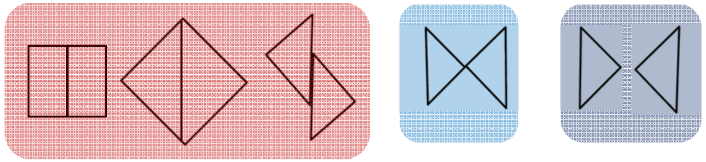
PLCA (1)

Symbolic representation for a spatial data focusing on connection patterns of regions

- Figures with the same connected patterns are regarded as the same

Originally designed to give a qualitative representation of a spatial object

- No numerical data

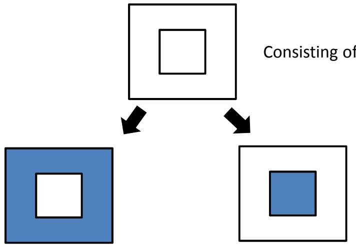


PLCA (2)

No pair of areas has a part in common

-- The entire figure is divided into disjoint areas

-- An area may have a hole



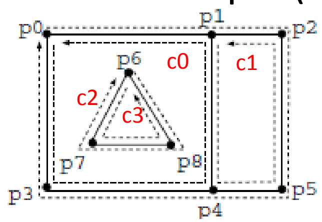
Data structure of PLCA

< P,L,C,A,o>

- P: a set of points
- L: a set of lines
  - a line is a pair of points (p_1,p_2), where each $p_i \in P$
  - each line l has a direction:
    $l^+=(p_1,p_2)$ ➔ $l^-=(p_2,p_1)$
- C: a set of circuits
  - a circuit is a list of lines [l_1,...,l_n] where each $l_i \in L$, $p_1=p_n$, where $l_i=(p_i,p_{i+1})$
- A: a set of areas
  - an area is a set of circuits {c_1,...,c_n} where each $c_i \in C$
- o: outermost, $o \in C$

Use **list** to implement most data structures

PLCA example (1)



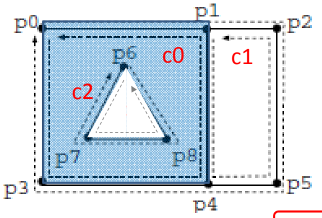
$$P = \{p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$$
$$L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9\}$$
$$C = \{o, c_0, c_1, c_2, c_3\}$$
$$A = \{a_0, a_1, a_2\}$$

$$l_0 = (p_0, p_1)$$
$$l_1 = (p_1, p_2)$$
$$l_2 = (p_0, p_3)$$
$$l_3 = (p_1, p_4)$$
$$l_4 = (p_2, p_5)$$
$$l_5 = (p_3, p_4)$$
$$l_6 = (p_4, p_5)$$
$$l_7 = (p_6, p_7)$$
$$l_8 = (p_7, p_8)$$
$$l_9 = (p_8, p_6)$$

$$c_0 = [l_2^+, l_5^+, l_3^-, l_0^-]$$
$$c_1 = [l_3^+, l_6^+, l_4^-, l_1^-]$$
$$c_2 = [l_9^-, l_8^-, l_7^-]$$
$$c_3 = [l_7^+, l_8^+, l_9^+]$$
$$o = [l_0^+, l_1^+, l_4^+, l_6^-, l_5^-, l_2^-]$$

$$a_0 = \{c_0, c_2\}$$
$$a_1 = \{c_1\}$$
$$a_2 = \{c_3\}$$

PLCA example (2)



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$c_2 = [l_9^-, l_8^-, l_7^-]$
$c_3 = [l_7^+, l_8^+, l_9^+]$
$o = [l_0^+, l_1^+, l_4^+, l_6^-, l_5^-, l_2^-]$

$a_0 = \{c_0, c_2\}$
$a_1 = \{c_1\}$
$a_2 = \{c_3\}$

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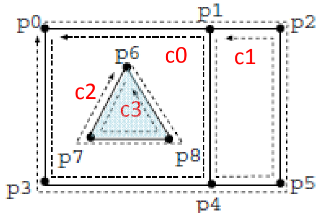
PLCA example (3)



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$c_3 = [l_7^+, l_8^+, l_9^+]$
$o = [l_0^+, l_1^+, l_4^+, l_6^-, l_5^-, l_2^-]$

$a_0 = \{c_0, c_2\}$
$a_1 = \{c_1\}$
$a_2 = \{c_3\}$

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Equivalence on PLCA expressions (1)

Sensitive on the equivalence of PLCA expressions in its implementation

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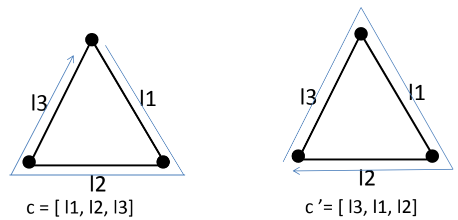
Equivalence on PLCA expressions (2)

Different expressions may stand for the same object in a symbolic expression



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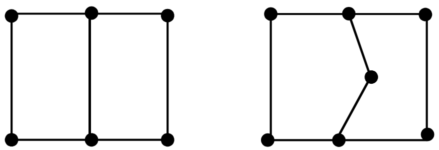
Equivalence on PLCA expressions (3)

The same subdivision should be handled differently in the symbolic expression



Equivalence on PLCA expressions (4)

Equivalence relation over PLCA expressions

P, L, C and A are equivalent to their permutations, respectively

Each element of L is equivalent to its inverse

Each element of C is equivalent to its rotation

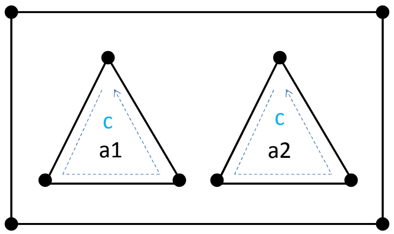
Each element of A is equivalent to its permutation

Consistent PLCA (1)

To permit only reasonable expressions

- putting topological constraint
- avoiding duplication in a list

a1 = { c }, a2 = { c }



Consistent PLCA (2)

Definition 3 (consistent PLCA). *Let $\langle P, L, C, A, o \rangle$ be a PLCA expression. The PLCA expression is said to be a* consistent PLCA expression *iff it satisfies all of the following conditions.*

- *For each point $p \in P$, there exists $l \in L$ such that $p \in l$.*
- *For each line $l \in L$, all points in l are in P.*
- *For each line $l \in L$, there exist circuits $c, c' \in C$ such that $l^+ \in c$ and $l^- \in c'$[3].*
- *For each circuit $c \in C$, each line in c or its inverse line is in L.*
- *For each circuit $c \in C$ except for o, there exists an area $a \in A$ such that $c \in a$.*
- *For each area $a \in A$, all circuits in a are in C.*
- *For any area $a \in A$, $o \notin a$.*
- *Each point $p \in P$ appears only once in P.*
- *Each line $l^+ \in L$ appears only once in L and $l^- \notin L$.*
- *Each circuit $c \in C$ appears only once in C and any rotated circuits other than c do not appear in C.*
- *Each area $a \in A$ appears only once in A and any equivalent area other than a does not appear in A.*

Consistent PLCA (3)

There does not exist more than one area that contains the same circuit

- For each point $p \in P$, there exists $l \in L$ such that $p \in l$.
- For each line $l \in L$, all points in l are in P.
- For each line $l \in L$, there exist circuits $c, c' \in C$ such that $l^+ \in c$ and $l^- \in c'$[3].
- For each circuit $c \in C$, each line in c or its inverse line is in L.
- For each circuit $c \in C$ except for o, there exists an area $a \in A$ such that $c \in a$.
- For each area $a \in A$, all circuits in a are in C.
- For any area $a \in A$, $o \notin a$.
- Each point $p \in P$ appears only once in P.
- Each line $l^+ \in L$ appears only once in L and $l^- \notin L$.
- Each circuit $c \in C$ appears only once in C and any rotated circuits other than c do not appear in C.
- Each area $a \in A$ appears only once in A and any equivalent area other than a does not appear in A.

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Outline

1. The target issue
2. A PLCA expression
3. Encoding PLCA in Coq
4. Representation of subdivision using PLCA
5. Conclusion

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Computational model of PLCA

- Inductively construct a PLCA expression using four constructors
- Intuitively,
  - Each constructor divides an existing area
- Symbolically,
  - It deletes the existing one area and adds two new areas, and reconfigures all the related objects

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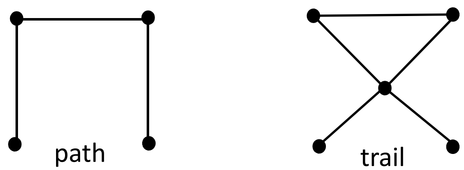
Data structure: path and trail

- A list of lines [$(p_0,p_1),(p_1,p_2),\ldots,(p_{n-1},p_n)$] (n>0)
- path: $p_0,\ldots,p_n$ are distinct

  used as a divider of a circuit to make a new area
- trail: $(p_0,p_1),\ldots,(p_{n-1},p_n)$ are distinct

  p0,…,pn may not be distinct

  used to show a property of a segment of a circuit



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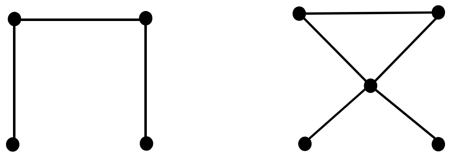
Closed path and closed trail (1)

Make a closed path/trail by adding a line connecting their start and end points



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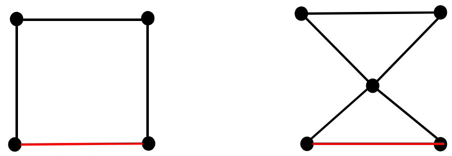
Closed path and closed trail (2)

Make a closed path/trail by adding a line connecting their start and end points

↓

circuit



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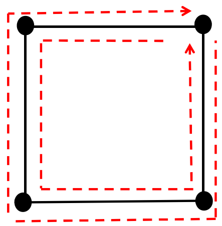
Example of closed path and closed trail (1)



(a)

- - - - closed path
- - - - closed trail

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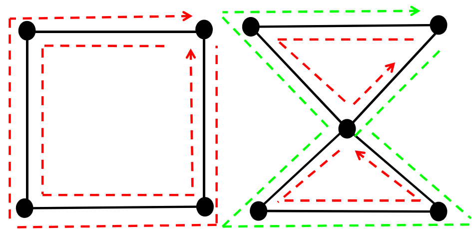
Example of closed path and closed trail (2)



(a) (b)

- - - - closed path
- - - - closed trail

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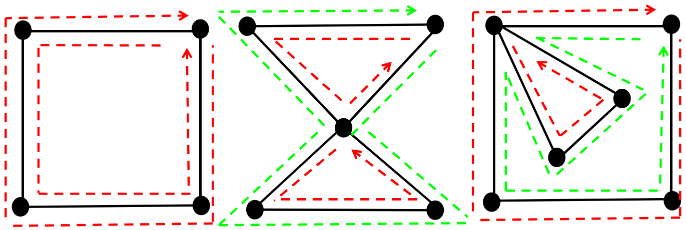
Example of closed path and closed trail (3)



(a) (b) (c)

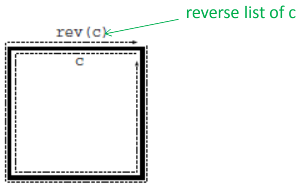
- - - - closed path
- - - - closed trail

Construction of a PLCA expression

- Using four constructors
  - single_loop
  - add_loop
  - add_inpath
  - add_inline

Constructor: single_loop

Make a Jordan curve as an outermost



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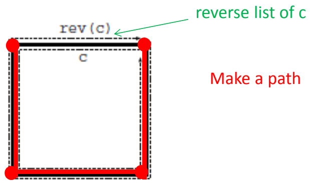

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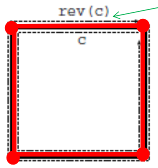
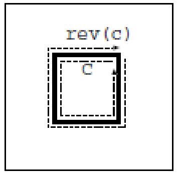


reverse list of c

Make a path
And connect its start and end points

Constructor: add_loop

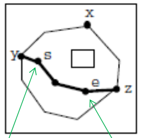
Add a Jordan curve in the specified area



Similar construction with that of single_loop

Constructor: add_inpath

Divide an area by cutting a specific circuit
At least two lines are added

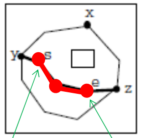


start point

end point

Constructor: add_inpath

Divide an area by cutting a specific circuit
At least two lines are added



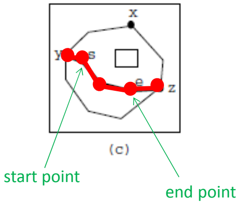
Make a path

start point

end point

Constructor: add_inpath

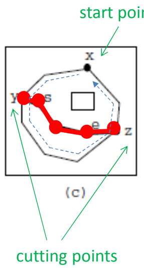
Divide an area by cutting a specific circuit
At least two lines are added



Make a path
And connect its start and end point
to the specified points in the same
existing circuit, respectively

Constructor: add_inpath

Divide an area by cutting a specific circuit
At least two lines are added

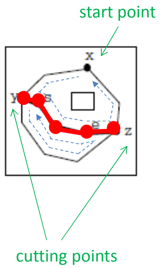


Make a path
And connect its start and end point
to the specified points in the same
existing circuit, respectively

Circuit has a cyclic structure

Constructor: add_inpath

Divide an area by cutting a specific circuit
At least two lines are added



Make a path
And connect its start and end point
to the specified points in the same
existing circuit

Circuit has a cyclic structure

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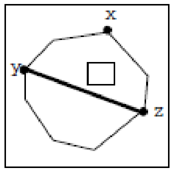
Reconfiguration of a circuit should begin with
the original start point

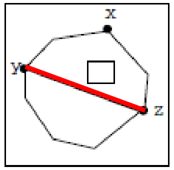
Constructor: add_inline

Make a short cut between two points of an existing circuit
Only one line is added



Constructor: add_inline

Make a short cut between two points of an existing circuit
Only one line is added



Make a line between the specified points
in the same existing circuit

Construction of a PLCA expression

- Using four constructors
  - single_loop
  - add_loop
  - add_inpath
  - add_inline
- Inductive PLCA (IPLCA)

  A PLCA expression constructed in this way

Outline

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IPLCA and surface subdivision

To show:

IPLCA represents a surface subdivision of a finite 2D plane that allows a region with holes

⇩

Represent the conditions for a subdivision

PLCA planarity

PLCA planarity

- The conditions for a surface subdivision
- Consists of three properties
  - PLCA-constraints
  - PLCA-connectedness
  - PLCA-euler

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PLCA-constraints

- Only the target figure can be expressed
  - No isolated points
  - No bridge between points
  - A finite plane encircled with a Jordan curve
  - Appropriate treatment of a duplicated point on a circuit

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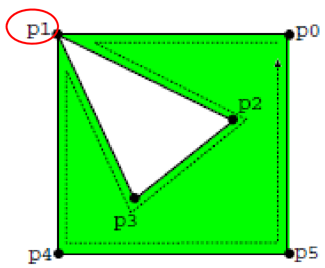
Treatment of a duplicated point



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PLCA-connectedness

- No objects including outermost are separated
  - Holding on different types of P,L,C,A
  - Conversion from these data types to a common data type and introduction of a new incidence function

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PLCA-euler

- Can be embedded in a 2D plane
  - An equality of numbers of the objects P,L,C,A and easily proved

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Planar PLCA

A planar PLCA

A consistent PLCA expression that satisfies PLCA-constraints, PLCA-connectedness and PLCA-euler

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Theorems on planarity

Theorem 1 [planarity]

Inductive PLCA is a planar PLCA.

Theorem 2 [planarity preservation]

Let *e,e'* be equivalent PLCA expressions.

If *e* is a planer PLCA, then *e'* is also a planar PLCA.

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Formal definition of a subdivision with PLCA

For each line,

if it is not included in the outermost,

- then distinct two areas are connected by the line

otherwise,

- there exists only one area that is connected with the outside of the figure by the line

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Theorem on subdivision

Theorem 3 [subdivision]

A planar PLCA is a subdivision of a finite 2D plane allowing a region with holes


```
Theorem PLCAsubdivision_area :
forall(P : list Point)(L : list Line)(C : list Circuit)(A : list Area)
  (o : Circuit),
 PLCA_planar P L C A o ->
 forall (l : Line),
  In l L
  -> ~LIn l o
  -> exists (c c' : Circuit),
   In c C
   /\ In c' C
   /\ In l c
   /\ In (reverse l) c'
   /\ exists (a a' : Area),
       In a A
       /\ In a' A
       /\ In c a
       /\ In c' a'
       /\ a <> a'.
```

subdivision: case1

```
Theorem PLCAsubdivision_outermost :
forall(P : list Point)(L : list Line)(C : list Circuit)(A : list Area)
  (o : Circuit),
 PLCA_planar P L C A o ->
 forall (l : Line),
  LIn l o
  -> exists (c : Circuit),
   In c C
   /\ LIn l c
   /\ o <> c
   /\ exists (a : Area),
       In a A
       /\ In c a.
```

subdivision: case2

Theorem on subdivision

Theorem 3 [subdivision]

A planar PLCA is a subdivision of a finite 2D plane allowing a region with holes

IPLCA represents a surface subdivision of a finite 2D plane that allows a region with holes

Theorem on subdivision

Theorem 3 [subdivision]

A planar PLCA is a subdivision of a finite 2D plane allowing a region with holes

IPLCA represents a surface subdivision of a finite 2D plane that allows a region with holes

Surface subdivision is formalized with PLCA and proved using Coq

Outline

1. The target issue
2. A PLCA expression
3. Encoding PLCA in Coq
4. Representation of subdivision using PLCA
5. Conclusion

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Conclusion

- Show a symbolic expression with PLCA for a surface subdivision of a finite 2D plane that allows holes in a region
- All formalization and proofs are performed using a proof assistant Coq (about 12,800 lines)
- Provide a computational model of PLCA
  - Give a certified proof
  - Relate QSR to software science

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Future works

- To revise implementation in the form not using list
- To implement high level computational geometry algorithms in PLCA and prove them
- To extend the approach to handle more general surface with an arbitrary genus

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Thank you!

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