An equivalence proof between rank theory and incidence projective geometry

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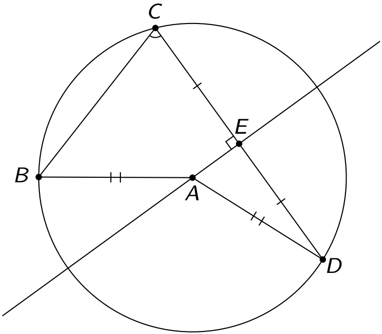
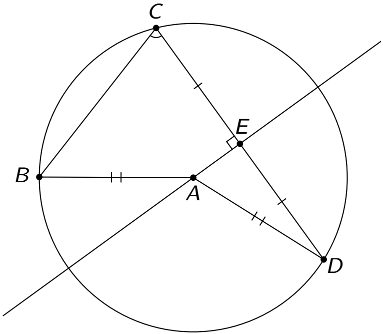
Euclidean Geometry



Geometric concepts

- Points
- Lines
- Incidence
- Distance
- Angles
- Circles
- ...

Euclidean Geometry



Geometric concepts

- **Points**
- **Lines**
- **Incidence**
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- ...

Incidence Geometry (IG)

Informal axiomatization of IG

1. There is always a line passing through two points
2. On any line, there are at least two points
3. There exist three points that are not aligned

Incidence Projective Geometry (PG)

Informal axiomatization of PG

1. There is always a line passing through two points
2. On any line, there are at least two points
3. There exist three points that are not aligned
4. **Two lines always meet in the plane**

Objective

Challenge

Establish an efficient procedure for decision to prove theorems of incidence projective geometry

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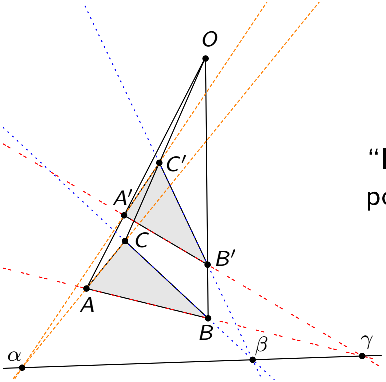
Goals

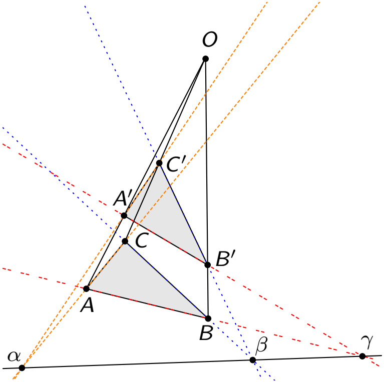
- Automation of proofs
- Prove theorems of projective geometry

Desargues Theorem



"If two triangles are perspective from a point, they are perspective from a line"

Desargues Theorem in PG



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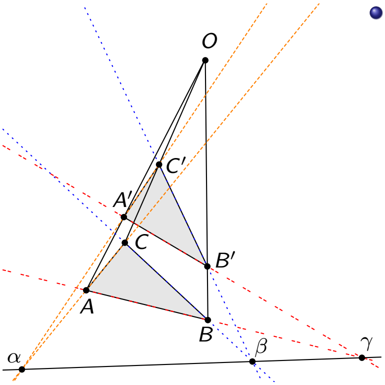
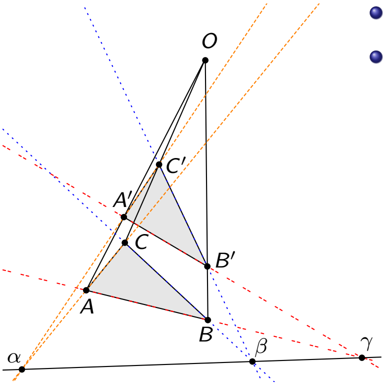


- **Ten points**: O A B C A' B' C' α β γ

Desargues Theorem in PG



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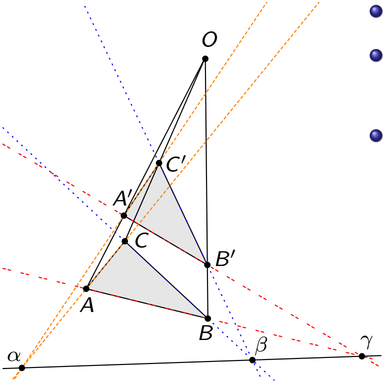
Desargues Theorem in PG



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- **Thirty incidences**: Incid O (OB), Incid B' (OB), Incid B (OB) ...

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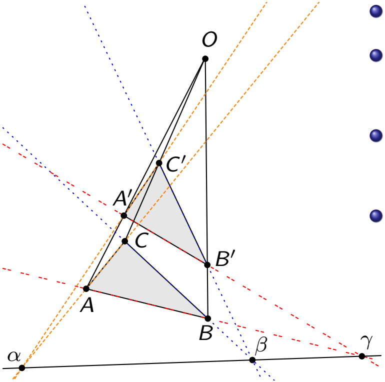

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- **Concepts**: equality, collinearity, coplanarity

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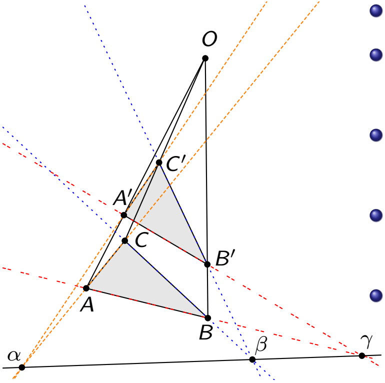

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- **Concepts**: equality, collinearity, coplanarity
- **Some conditions to deal with degenerate cases**

Rank theory (RK)

Concept of rank

Integer function noted $rk(\mathrm{E})$ returning the dimension of the set of points E

Rank theory (RK)

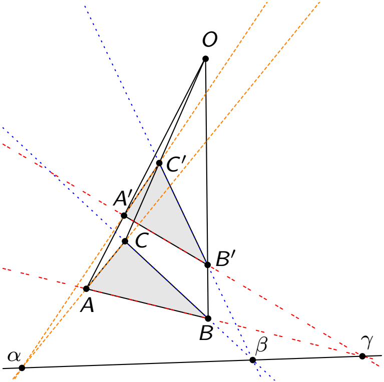
Concept of rank

Integer function noted $rk(\mathrm{E})$ returning the dimension of the set of points E

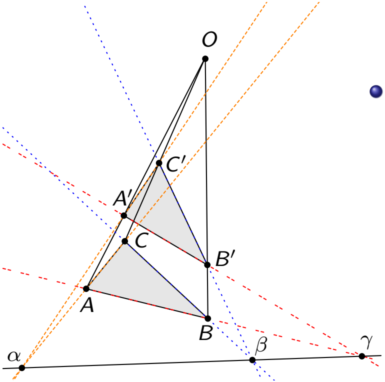
Some examples

$rk\{A,B\} = 1$	A = B
$rk\{A,B\} = 2$	$A \neq B$
$rk\{A,B,C\} = 2$	A,B,C are collinear with at least two of them distinct
$rk\{A,B,C\} \leq 2$	A,B,C are collinear
$rk\{A,B,C\} = 3$	A,B,C are not collinear
$rk\{A,B,C,D\} = 3$	A,B,C,D are coplanar, not all collinear
$rk\{A,B,C,D\} = 4$	A,B,C,D are not coplanar

Desargues Theorem in RK



Desargues Theorem in RK



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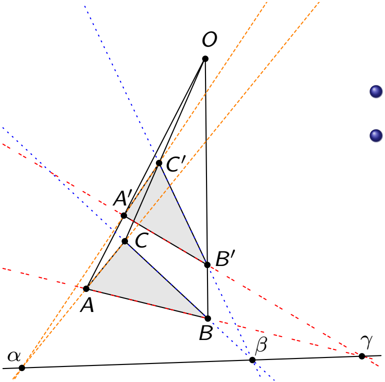
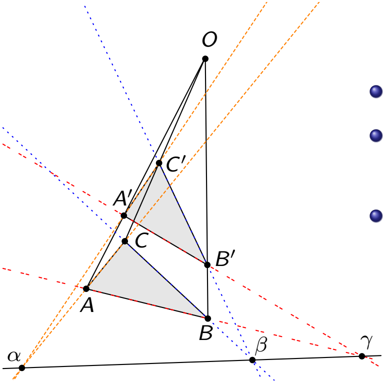
Desargues Theorem in RK



- **Ten points**: O A B C A' B' C' α β γ
- **Ten sets**: rk\{O,A,A'\} = 2, rk\{A B γ\} = 2, $rk\{\alpha\ \beta\ \gamma\} = 2$ ...

Desargues Theorem in RK



- **Ten points**: O A B C A' B' C' α β γ
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Establish an efficient procedure for decision **based on the notion of rank** to prove theorems of incidence projective geometry

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Establish an efficient procedure for decision **based on the notion of rank** to prove theorems of incidence projective geometry

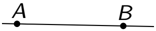
Goals

- **Prove the equivalence between the two approaches**
- **Develop a bilateral process of translation**
- Automation of proofs
- Prove theorems of projective geometry

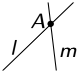
Axioms of projective geometry in 2D

5 axioms

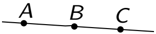
- **Line-Existence** : $\forall$ A B : Point, $\exists$ l : Line, A $\in$ l $\wedge$ B $\in$ l



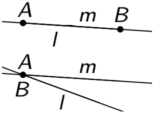
- **Point-Existence** : $\forall$ l m : Line, $\exists$ A : Point, A $\in$ l $\wedge$ A $\in$ m



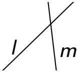
- **Three-Points** : $\forall$ l : Line, $\exists$ A B C : Point, A $\neq$ B $\wedge$ B $\neq$ C $\wedge$ A $\neq$ C $\wedge$ A $\in$ l $\wedge$ B $\in$ l $\wedge$ C $\in$ l



- **Uniqueness** : $\forall$ A B : Point, $\forall$ l m : Line, A $\in$ l $\wedge$ B $\in$ l $\wedge$ A $\in$ m $\wedge$ B $\in$ m $\Rightarrow$ A $=$ B $\vee$ l $=$ m



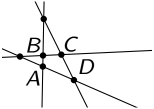
- **Lower-Dimension-2** : $\exists$ l m : Line, l $\neq$ m



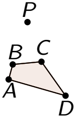
Axioms of projective geometry in 3D

5 axioms

- **Line-Existence** : ∀ A B : Point, ∃ l : Line, $A \in l \wedge B \in l$
- **Pasch** : ∀ A B C D : Point, ∀ l_{AB} l_{CD} l_{AC} l_{BD} : Line, $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$, $A \in l_{AB} \wedge B \in l_{AB} \wedge C \in l_{CD} \wedge D \in l_{CD} \wedge$ $A \in l_{AC} \wedge C \in l_{AC} \wedge B \in l_{BD} \wedge D \in l_{BD} \wedge$ $(\exists I : \text{Point}, I \in l_{AB} \wedge I \in l_{CD}) \Rightarrow$ $(\exists J : \text{Point}, J \in l_{AC} \wedge J \in l_{BD})$



- **Three-Points** : ∀ l : Line, ∃ A B C : Point, $A \neq B \wedge B \neq C \wedge A \neq C \wedge A \in l \wedge B \in l \wedge C \in l$
- **Uniqueness** : ∀ A B : Point, ∀ l m : Line, $A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow A = B \vee l = m$
- **Lower-Dimension-3** : ∃ l m : Line, ∀ p : Point, $p \notin l \vee p \notin m$



Matroid theory

Origin

The rank function is one of the fundamental concepts of matroid theory

Multiple applications

Some application fields: graph theory, greedy algorithm, geometric configuration, linear algebra, combinatorial optimization

Several axiomatizations

Axiomatization around the concepts: independent sets, bases, circuits, rank function, closure operation, flat ...

Rank

Rank

The integer function *rk* on a set E is the rank function associated to a matroid iff:

- **R1** : $\forall$ X $\subseteq$ E, $0 \leq rk(X) \leq |X|$
  (non negative and subcardinal)
- **R2** : $\forall$ X Y $\subseteq$ E, X $\subseteq$ Y $\Rightarrow rk(X) \leq rk(Y)$
  (non decreasing)
- **R3** : $\forall$ X Y $\subseteq$ E, $rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$
  (submodular)

Axioms of ranks in 3D

8 axioms

- **R1 + R2 + R3**
- **Rk-singleton** : $\forall$ P, $rk\{P\} \geq 1$
- **Rk-couple** : $\forall$ P Q, P $\neq$ Q $\Rightarrow$ $rk\{P,Q\} \geq 2$
- **Rk-Pasch** : $\forall$ A B C D,
  $rk\{A,B,C,D\} \leq 3 \Rightarrow \exists$ J, $rk\{A,B,J\} = rk\{C,D,J\} = 2$
- **Rk-Three-Points** : $\forall$ A B,
  $\exists$ C, $rk\{A,B,C\} = rk\{B,C\} = rk\{A,C\} = 2$
- **Rk-Lower-Dimension** : $\exists$ A B C D, $rk\{A,B,C,D\} \geq 4$

Equivalence

Theorem

Axiomatization on incidence projective geometry and rank-based axioms system are equivalent respectively in 2D and 3D

Equivalence

Theorem

Axiomatization on incidence projective geometry and rank-based axioms system are equivalent respectively in 2D and 3D

Dimension and direction

1. Axiomatization of PG in 2D $\Longleftarrow$ Axiomatization of RK in 2D
2. Axiomatization of PG in 3D $\Longleftarrow$ Axiomatization of RK in 3D
3. Axiomatization of PG in 2D $\Longrightarrow$ Axiomatization of RK in 2D
4. Axiomatization of PG in 3D $\Longrightarrow$ Axiomatization of RK in 3D

Theoretical details

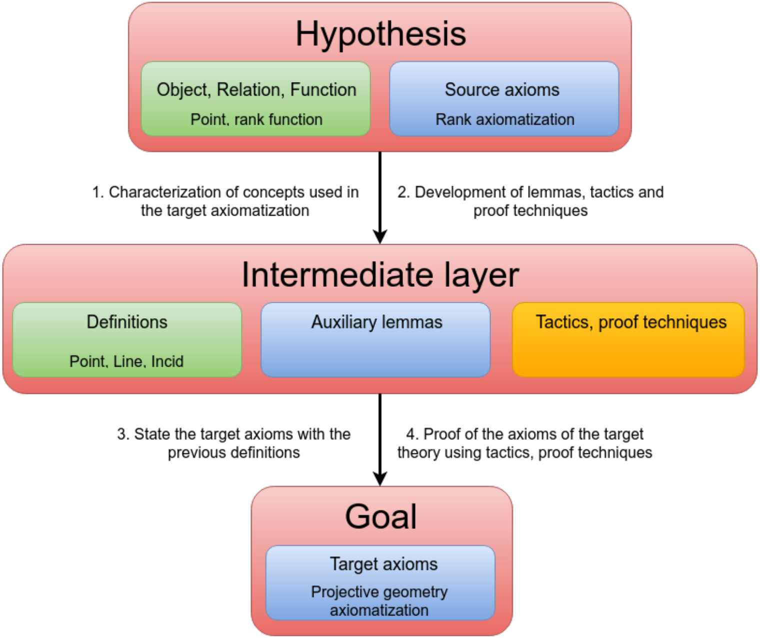
Decidability issue

Decidability of incidence — Incid a m $\vee$ $\neg$Incid a m

- Decidability of points equality — $a = b \vee a \neq b$
- Decidability of lines equality — $l = m \vee l \neq m$

Equality issue

- Parametric equality for points — a[==]b
- Classical equality for lines — $l = m$


Hypothesis
Object, Relation, Function
Point, rank function
Source axioms
Rank axiomatization
1. Characterization of concepts used in the target axiomatization
2. Development of lemmas, tactics and proof techniques
Intermediate layer
Definitions
Point, Line, Incid
Auxiliary lemmas
Tactics, proof techniques
3. State the target axioms with the previous definitions
4. Proof of the axioms of the target theory using tactics, proof techniques
Goal
Target axioms
Projective geometry axiomatization

Overview

Some data

- 21 definitions
- 350 lemmas
- 150 tactics
- 15000 lines of Coq

	RK to PG		PG to RK	
	2D	3D	2D	3D
Lines of Coq specs	250	350	650	1050
Lines of Coq proofs	300	800	2600	11000

Conclusion

Results

- Equivalence between two axiomatizations in both 2D & 3D
- Implementation of automation process

Current & future work

- Bilateral translation
- Automation proofs with ranks
- Other cryptomorphic axiomatizations on matroids

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One axiomatization of matroids

- A set is either independent or dependent.
- The empty set is independent.
- Subsets of an independent set are independent.
- If the sets U and V are independent, and if V has one more element than U, then it is possible to complete U with an element $v \in V\text{-}U$ such that $U \cup \{v\}$ is independent.

From RK to PG

Characterization

Definition of Point & Line

```
Definition Point := Point.

Inductive LineInd : Type :=
| Cline : forall (A B : Point)(H : ~A[==]B), LineInd
```

Definition of Incid

```
Definition Incid (P : point)(l : Line) :=
rk ((fstP l)(sndP l) P) = 2.
```

From RK to PG

Proof techniques

- **Rank equality**: $rk(\mathrm{a}) = rk(\mathrm{b}) \implies rk(\mathrm{a}) \geq rk(\mathrm{b}) \wedge rk(\mathrm{a}) \leq rk(\mathrm{b})$
- **Submodularity**: $rk(\mathrm{X} \cup \mathrm{Y}) + rk(\mathrm{X} \cap \mathrm{Y}) \leq rk(\mathrm{X}) + rk(\mathrm{Y})$

Proof principle

1. State axiom of projective geometry as a lemma
2. Substitute assertions with previous definitions
3. Use proof techniques and rank axioms to achieve the proof

From PG to RK

Characterization

Définition

```
Definition rk (m : set point) : nat :=
match m with
| empty ⇒ 0
| add x empty ⇒ 1
| m ⇒ if contains_three_non_collinear_points m then 3
       else if contains_two_distinct_points m then 2
       else 1
end.
```

From PG to RK

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- **Either the set is empty**

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- Either the set is empty
- **Either the set represents a point**

From PG to RK

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- Either the set is empty
- Either the set represents a point
- **Either the set represents a line**

From PG to RK

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end.
```

- Either the set is empty
- Either the set represents a point
- Either the set represents a line
- **Either the set represents a plane**

From PG to RK

Structural induction issue

(R2): $\forall$ X $\subseteq$ Y, $rk(X) \leq rk(Y)$

- $0 \leq 0 \Rightarrow \emptyset \subset \emptyset$
- $1 \leq 3 \Rightarrow$ Point $\subset$ Plane
- $3 \leq 2 \Rightarrow$ Plane $\not\subset$ Line

Axiom of submodularity

(R3): $rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$

```
Lemma matroid3_rk2_rk2_interrk2_to_unionrk2 :
forall l m,
rk l = 2 →
rk m = 2 →
rk (l ∩ m) = 2 →
rk (l ∪ m) = 2.
```